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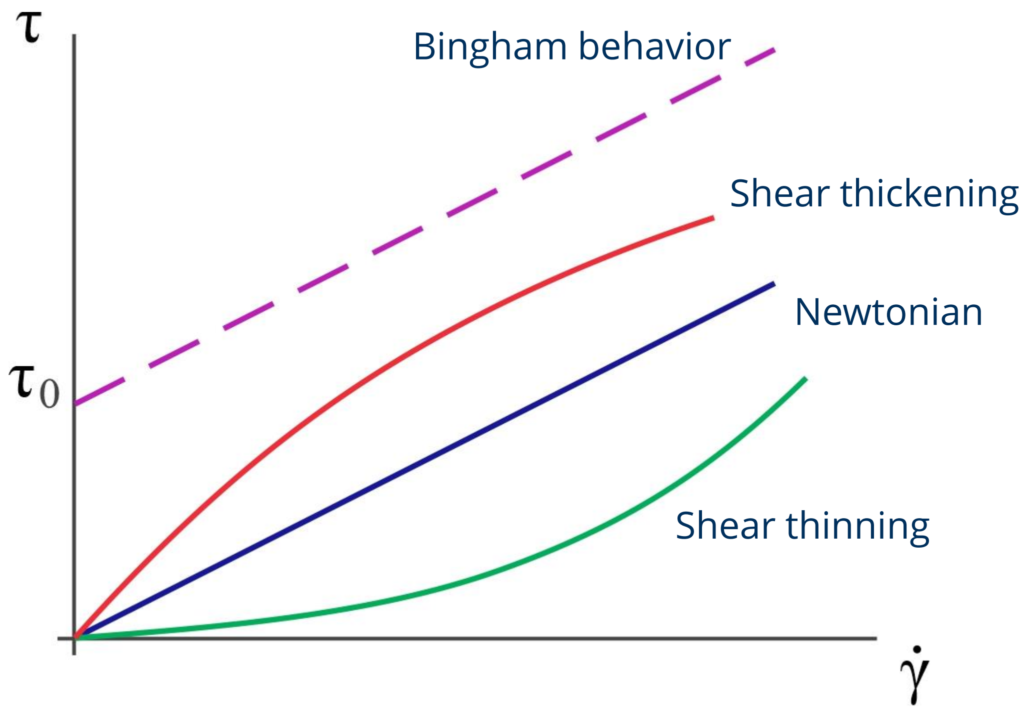
From Torque-Velocity Data to Local Rheology: Inversion –Based Recovery of Yield Stress and Shear-Rate Profiles

Rheology of Building Materials // 25.02.2026

Rheology characterization
Constitutive laws

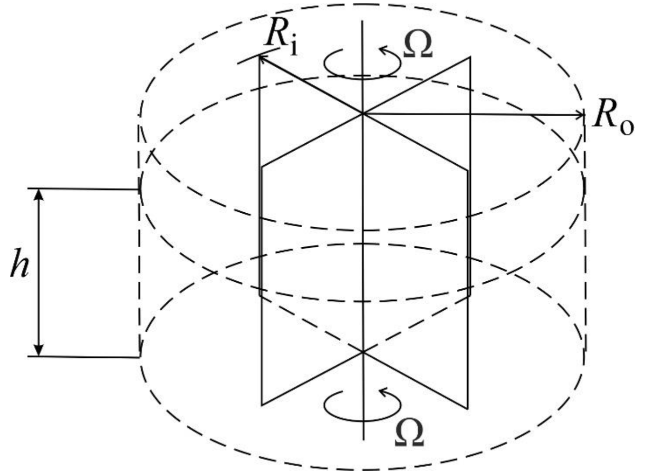


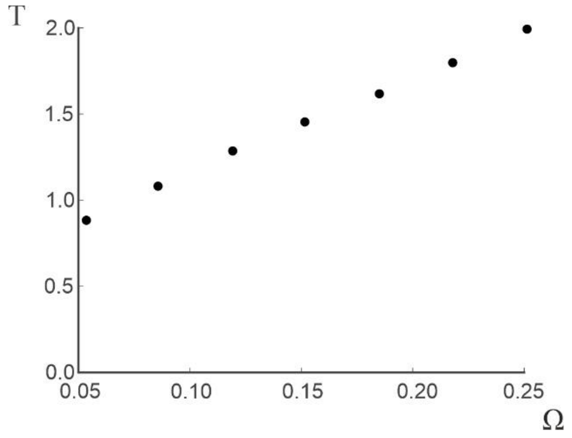
Meaningful for many technological processes

The constitutive law determines behavior

$$\tau = \tau_0 + f(\dot{\gamma})$$

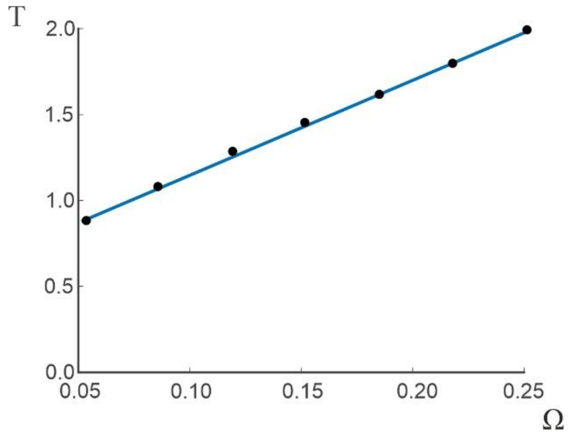
Rheology characterization
Practical approach





Model for $\tau = \tau_0 + f(\dot{\gamma})$

Linear $\tau = \tau_0 + \mu\dot{\gamma}$



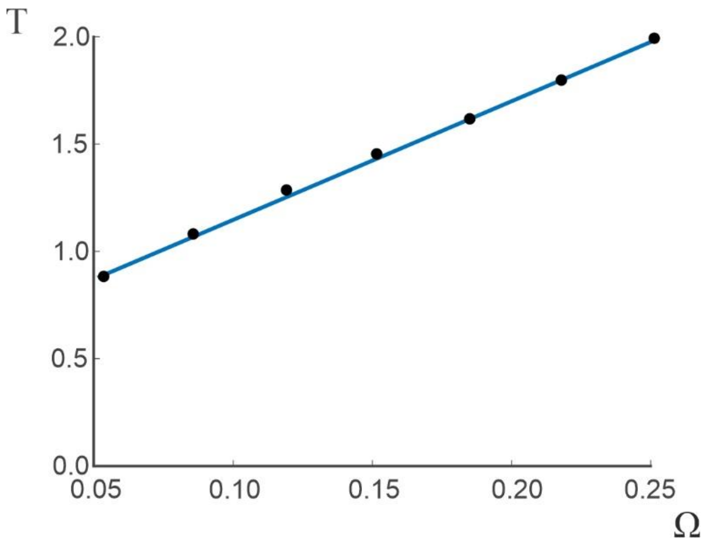
Fit - extract

τ_0, μ

Here and in the following: $[T] - Nm; [\Omega] - s^{-1}$

Rheology characterization
Practical approach

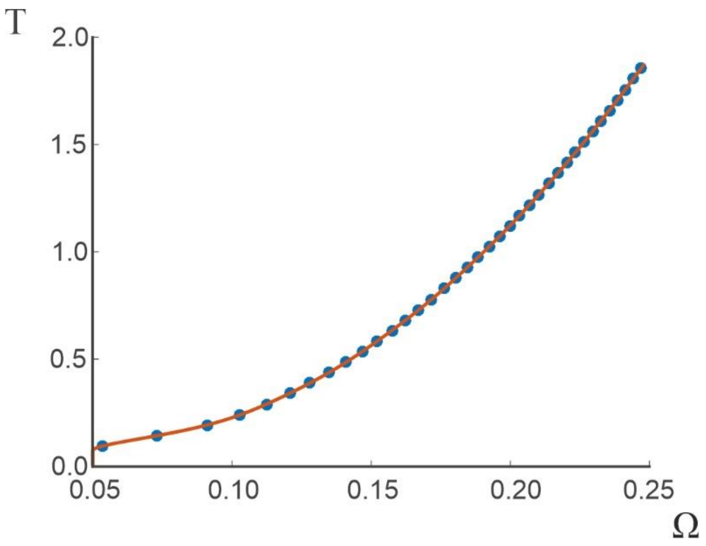
Linear case



$$\tau = \tau_0 + \mu\dot{\gamma}$$

Unique solution

Non -Linear case



$$\tau = \tau_0 + f(\dot{\gamma})$$

No unique solution

Different models (HB, MB, …)

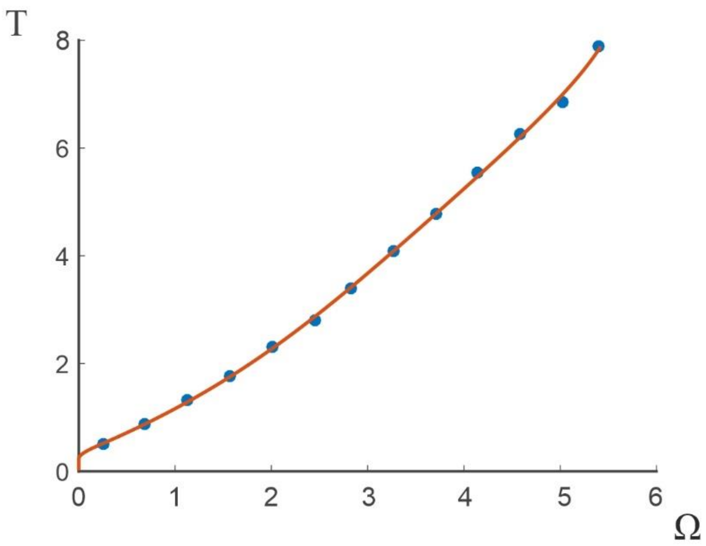
fit data equally well BUT – yields different parameters!

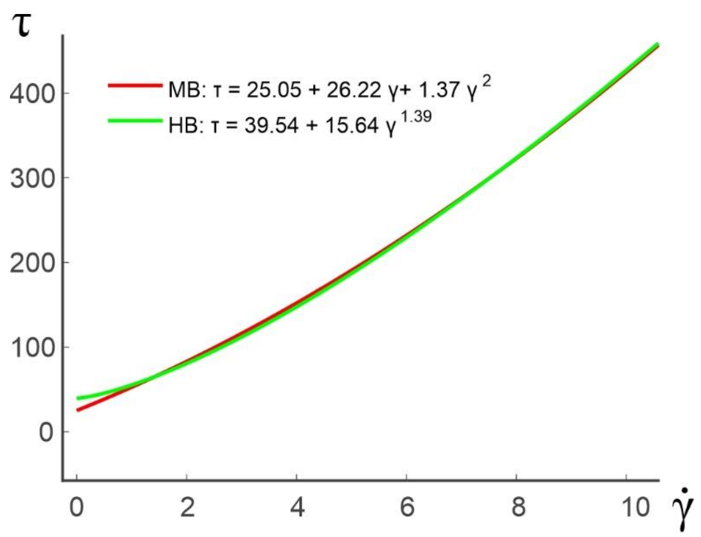
How to extract

$\tau_0, \mu, \ldots$

Rheology characterization
Practical approach

Example: Non -Linear case





$$\tau = \tau_0 + f(\dot{\gamma})$$

MB: $\tau = \tau_0 + \mu\dot{\gamma} + c\dot{\gamma}^2$ → 25 Pa

HB: $\tau = \tau_0 + k\dot{\gamma}^n$ → 40 Pa

Here and in the following: $[\tau] - Pa; [\dot{\gamma}] - s^{-1}$

Rheology characterization
Reiner – Riwlin transformation

Problem: find transformation from measured data - $[T, \Omega]$ (global) – to $[\tau, \dot{\gamma}]$ (local):

Corrections: plug flow, shear rate (Krieger-Elrod), inertia, etc.

Key models

- Bingham
- Modified Bingham
- Herschel-Bulkley
- Power-Law
- Hagen-Poiseuille
- …

Rheology characterization

Back to fundamentals

Textbook relations

$$\dot\gamma = r\frac{d\omega(r)}{dr} \quad \Rightarrow \quad \int_{R_i}^{R_o}\frac{\dot\gamma(r)}{r}dr = \int_0^{\Omega} d\omega = \Omega \longrightarrow \Omega(T)$$

$$\tau(r) = \frac{T}{2\pi h r^2} \qquad \xi = \frac{T}{2\pi h} \qquad \tau = \frac{\xi}{r^2}$$

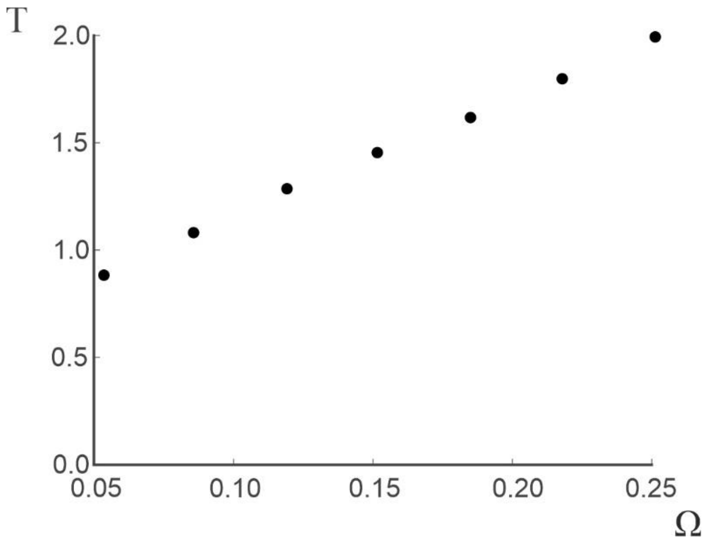
$$\tau = f(\dot\gamma) \Rightarrow \dot\gamma = f^{-1}(\tau) = g(\tau)$$

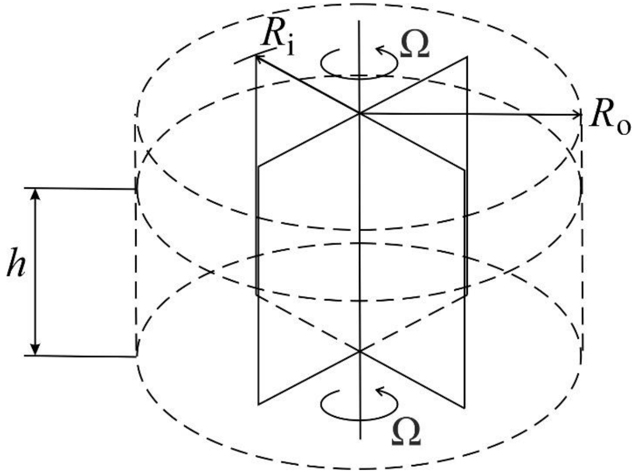
The most interesting

$$\Omega(\xi) = \int_{R_i}^{R_o}\frac{g(\xi/r^2)}{r}dr \Rightarrow g(\tau) = \boxed{\dot\gamma = 2\tau R_i^2 \sum_{k=0}^{\infty}\alpha^k \frac{d\Omega(\alpha^k \tau R_i^2)}{d\xi};} \quad \alpha = \frac{R_i^2}{R_o^2}$$

$$D(\xi) = \frac{d\Omega(\xi)}{d\xi}$$

[*F. R. De Hoog and R.S. Anderssen, Regularization of first kind integral equations with application to couette viscometry, Journal of integral equations and applications, 18 (2), 2006*]



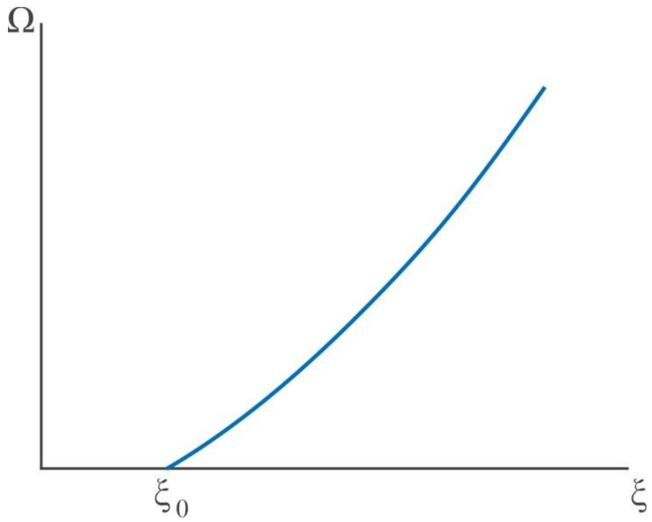


Rheology characterization
Back to fundamentals

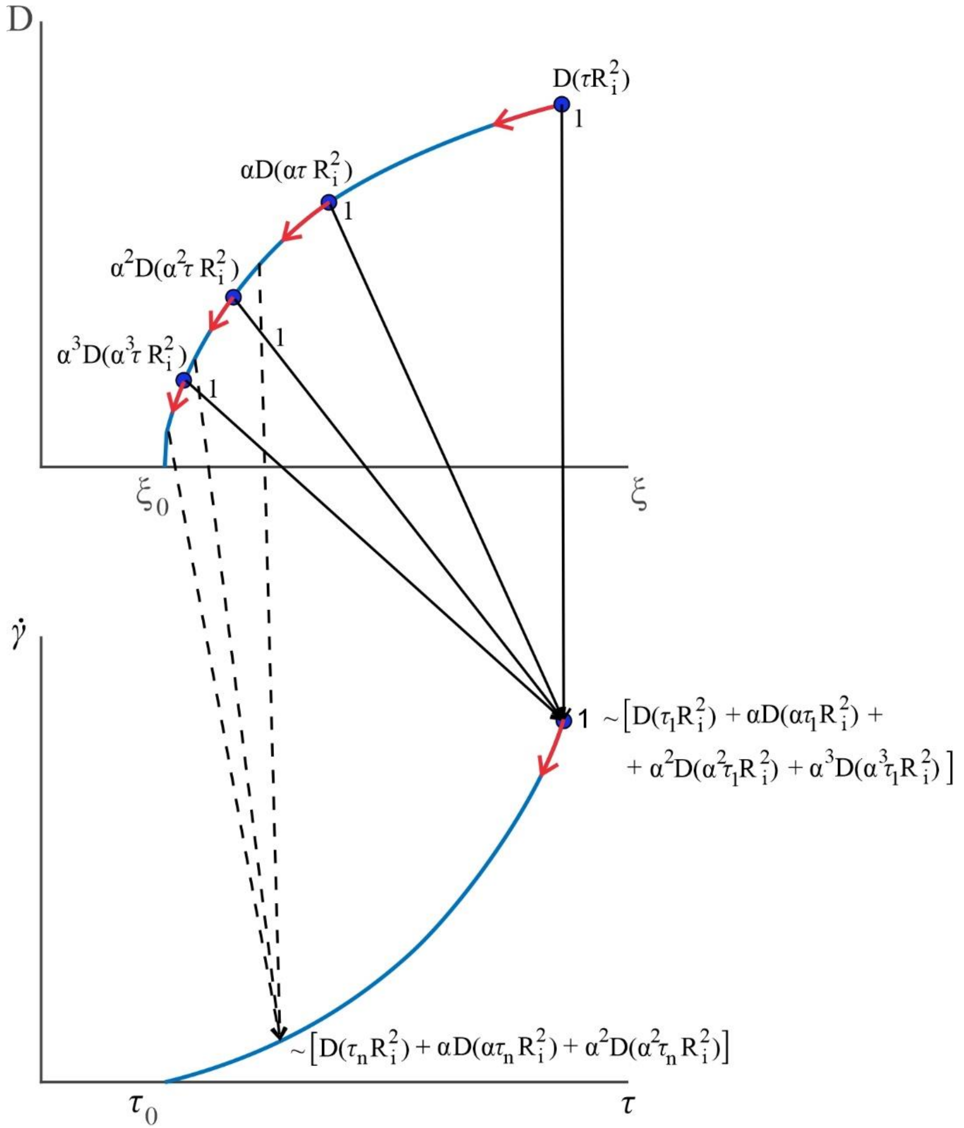
$$\dot{\gamma} = 2\tau R_i^2 \sum_{k=0}^{\infty} \alpha^k \frac{d\Omega(\alpha^k \tau R_i^2)}{d\xi} = 2\tau R_i^2 \sum_{k=0}^{\infty} \alpha^k D(\alpha^k \tau R_i^2); \quad \alpha = \frac{R_i^2}{R_o^2} < 1;$$

$$D(\xi) = \frac{d\Omega(\xi)}{d\xi}$$

$$\dot{\gamma} = 2\tau R_i^2 \left(D(\tau R_i^2) + \alpha D(\alpha \tau R_i^2) + \alpha^2 D(\alpha^2 \tau R_i^2) + \cdots \right)$$



Nonlocal effect

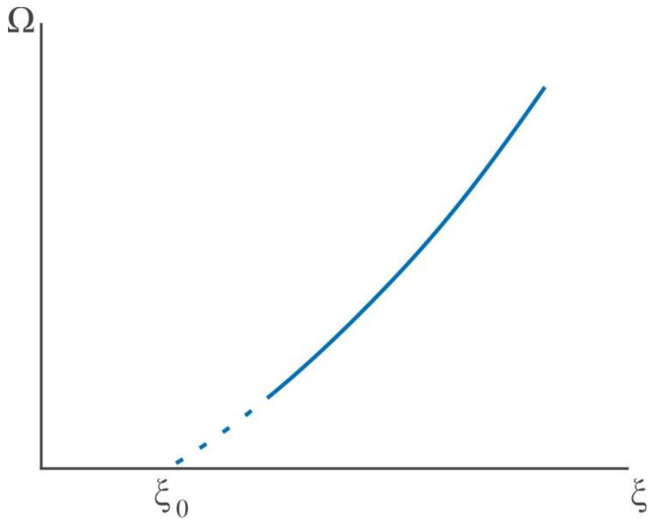


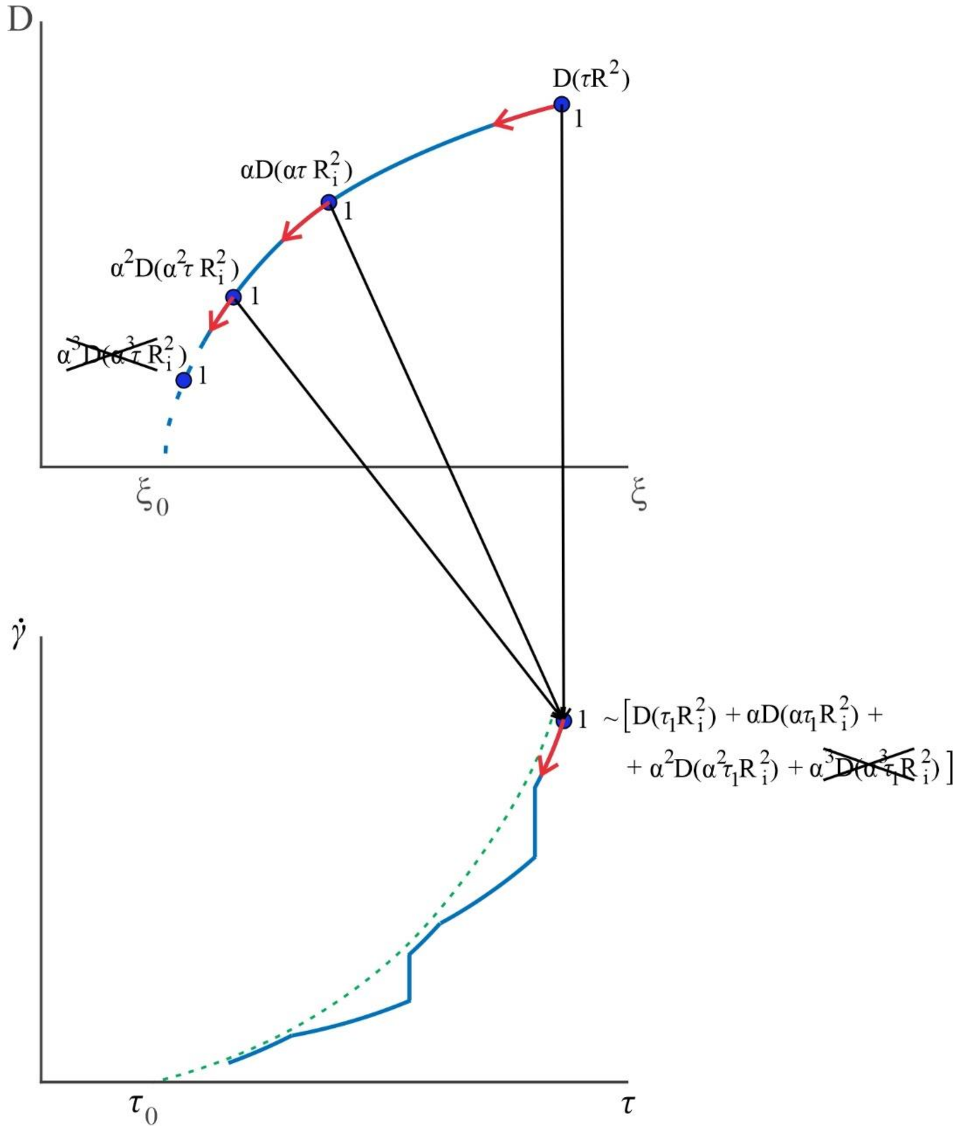
Rheology characterization
Reality

$$\dot{\gamma} = 2\tau R_i^2 \sum_{k=0}^{\infty} \alpha^k \frac{d\Omega(\alpha^k \tau R_i^2)}{d\xi} = 2\tau R_i^2 \sum_{k=0}^{\infty} \alpha^k D(\alpha^k \tau R_i^2)\,; \quad \alpha = \frac{R_i^2}{R_o^2} < 1;$$

$$D(\xi) = \frac{d\Omega(\xi)}{d\xi}$$

$$\dot{\gamma} = 2\tau R_i^2 \left(D(\tau R_i^2) + \alpha D(\alpha \tau R_i^2) + \alpha^2 D(\alpha^2 \tau R_i^2) + \cdots \right)$$



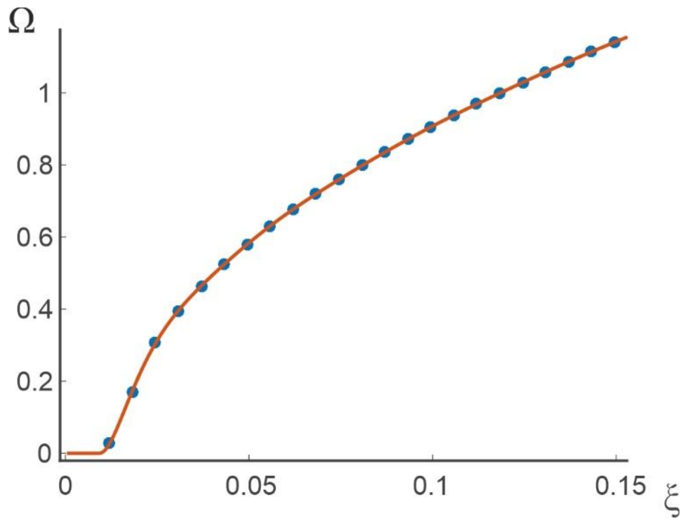


Rheology characterization
Back to fundamentals

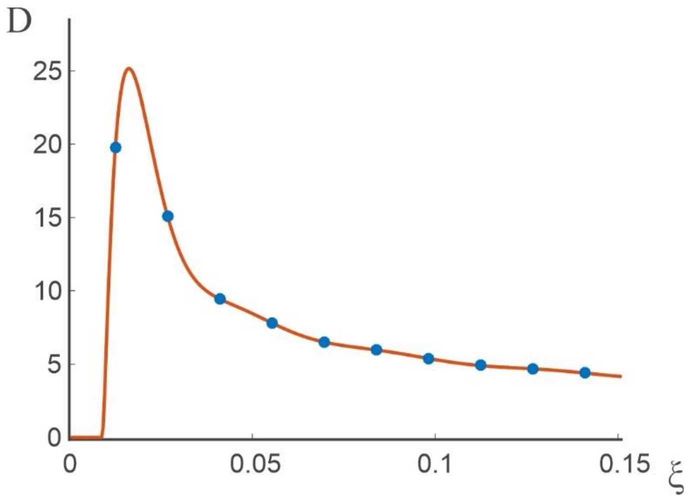
$$\dot{\gamma} = 2\tau R_i^2 \sum_{k=0}^{\infty} \alpha^k \frac{d\Omega(\alpha^k \tau R_i^2)}{d\xi} = 2\tau R_i^2 \sum_{k=0}^{\infty} \alpha^k D(\alpha^k \tau R_i^2); \quad \alpha = \frac{R_i^2}{R_o^2} < 1; \quad D(\xi) = \frac{d\Omega(\xi)}{d\xi}$$

$$\dot{\gamma} = 2\tau R_i^2 \left(D(\tau R_i^2) + \alpha D(\alpha \tau R_i^2) + \alpha^2 D(\alpha^2 \tau R_i^2) + \cdots \right)$$

Even when Ω is "normal"



D is "not"



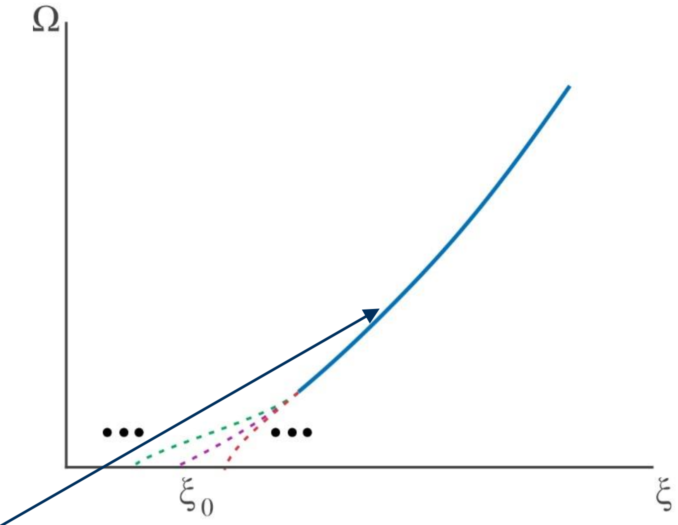
Rheology characterization

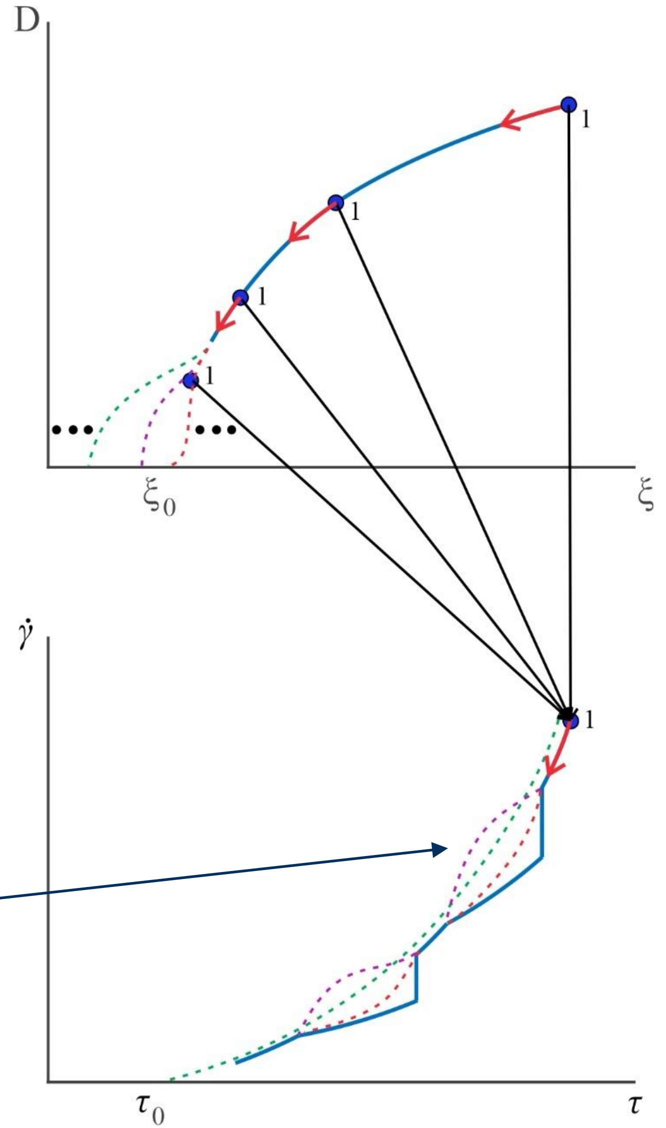
How to deal with it - physical constraints

For yield stress behaviour the expression is similar

$$\dot{\gamma}(\tau) = 2\,R_i^2(\tau + \tau_0)\sum_{k=0}^{\infty}\alpha^k \left.\frac{d\Omega(\xi_k)}{d\xi}\right|_{\xi=\xi_k},$$

$$\xi_k = \alpha^k R_i^2(\tau + \tau_0)$$

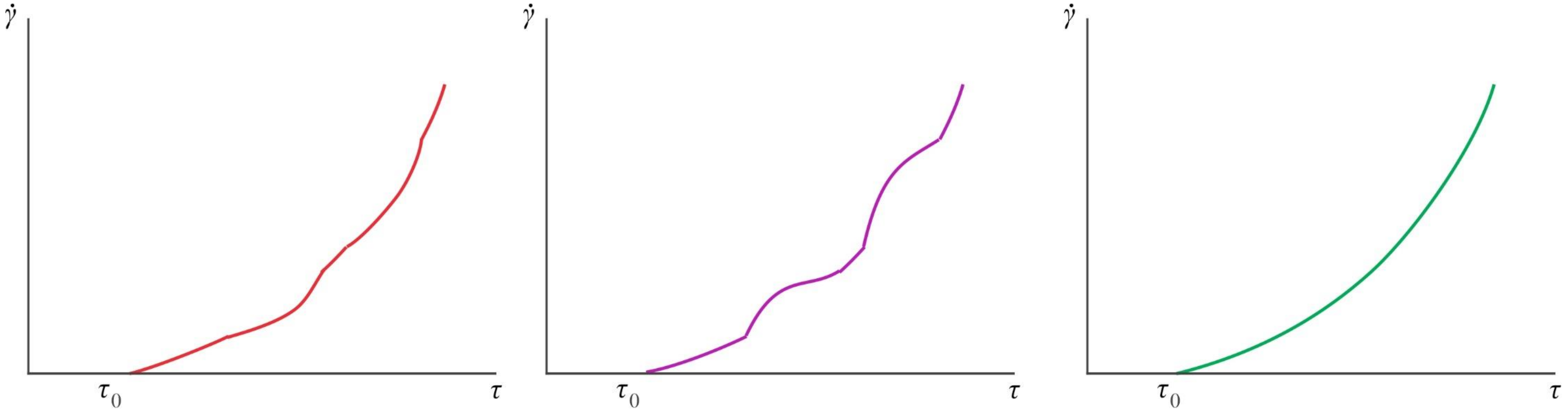




Optimization criteria: RMSE + SMOOTHNESS

Rheology characterization
Physical constraints

$$\dot{\gamma}(\tau) = 2\,R_i^2(\tau + \tau_0)\sum_{k=0}^{\infty}\alpha^k \left.\frac{d\Omega(\xi_k)}{d\xi}\right|_{\xi=\xi_k}, \qquad \xi_k = \alpha^k R_i^2(\tau + \tau_0)$$



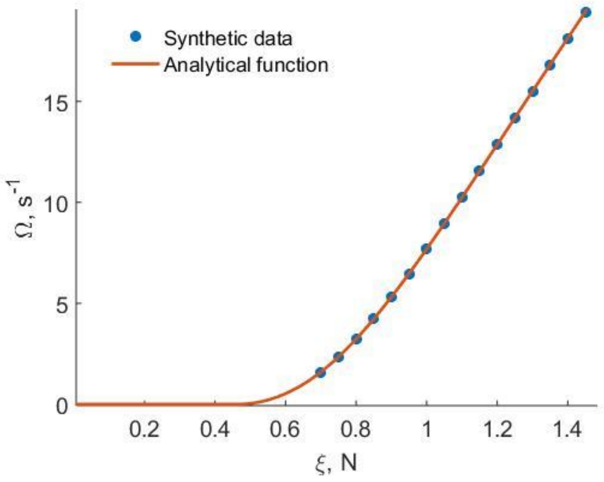
Optimization criteria: RMSE + SMOOTHNESS

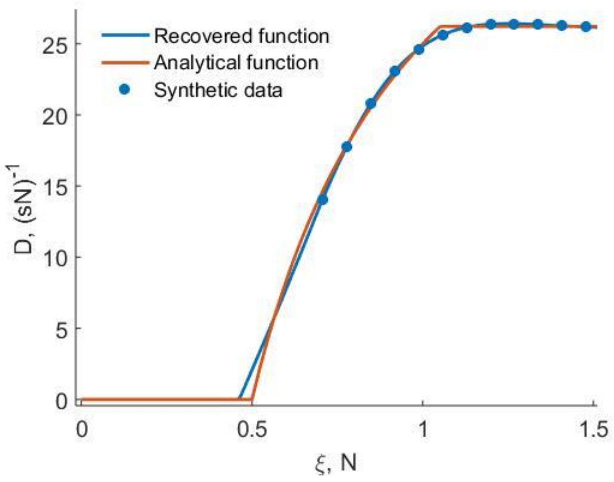
Rheology characterization
Example I

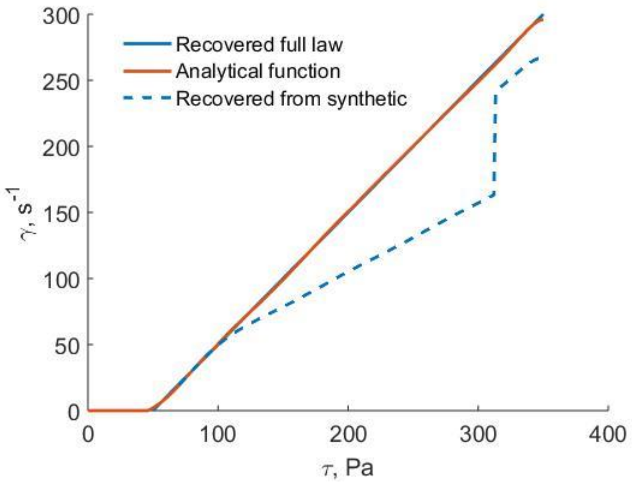
$$\tau = \tau_0 + \mu\dot{\gamma}$$

$$\Omega(\xi) = \begin{cases} \frac{\xi}{2}\left(\frac{1}{R_i^2} - \frac{1}{R_o^2}\right) - \frac{\tau_0}{2}\ln\frac{R_o^2}{R_i^2}, & \frac{\xi}{\tau_0} \geq R_o^2 \\ \frac{\xi}{2}\left(\frac{1}{R_i^2} - \frac{\tau_0}{\xi}\right) - \frac{\tau_0}{2}\ln\frac{\xi}{\tau_0 R_i^2}, & R_i^2 \leq \frac{\xi}{\tau_0} < R_o^2 \end{cases}$$

$\tau_0 = 50$ Pa, $\mu = 1$ Pa s, $R_i = 0.1$ m, $R_o = 0.145$ m.





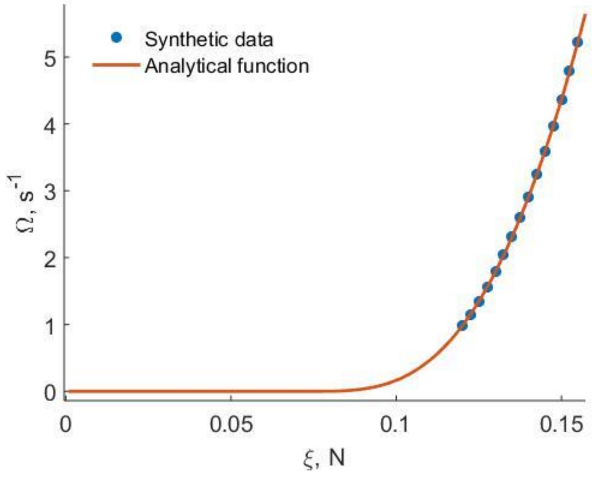


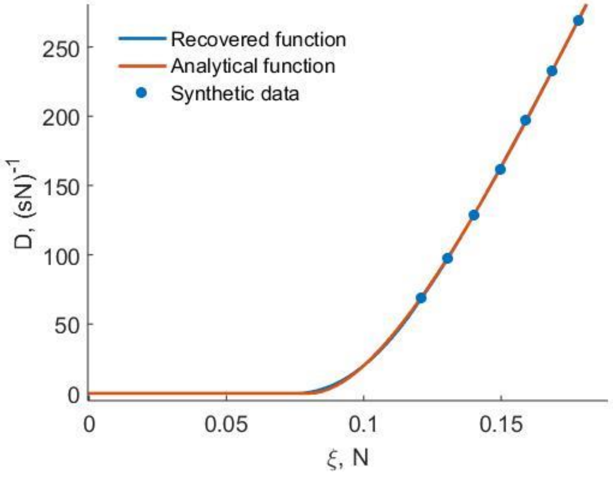
Rheology characterization
Example II

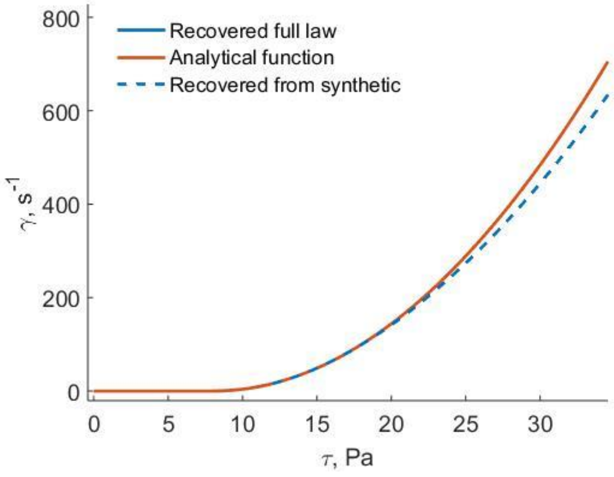
$$\tau = \tau_0 + \mu\dot{\gamma}^{1/2}$$

$$\Omega(\xi) = \begin{cases} \dfrac{\xi^2}{4}\left(\dfrac{1}{R_i^4} - \dfrac{1}{R_o^4}\right) + \xi\tau_0\left(\dfrac{1}{R_o^2} - \dfrac{1}{R_i^2}\right) + \tau_0^2 \ln\dfrac{R_o}{R_i}, & \dfrac{\xi}{\tau_0} \geq R_o^2 \\ \dfrac{1}{4}\left(\dfrac{\xi^2}{R_i^4} - \tau_0^2\right) + \left(\tau_0^2 - \dfrac{\xi\tau_0}{R_i^2}\right) + \dfrac{\tau_0^2}{2}\ln\dfrac{\xi/\tau_0}{R_i^2}, & R_i^2 \leq \dfrac{\xi}{\tau_0} < R_o^2 \end{cases}$$

$\tau_0 = 8.0$ Pa, $\mu = 1$ Pa s, $R_i = 0.1$ m, $R_o = 0.145$ m





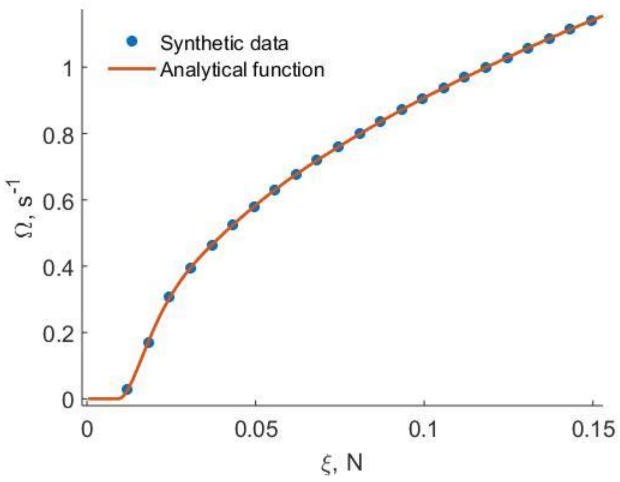


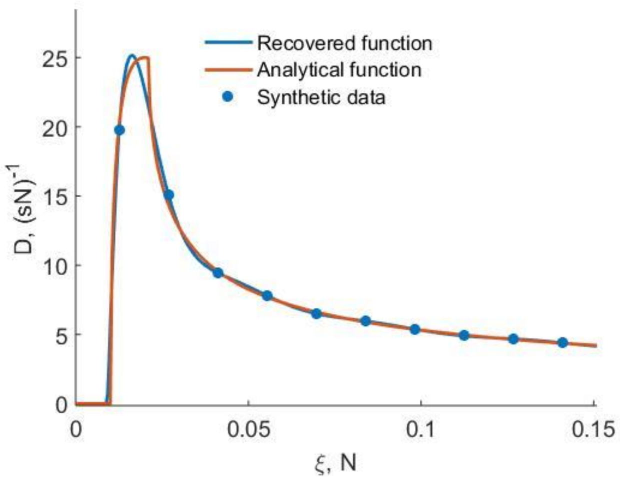
Rheology characterization
Example III

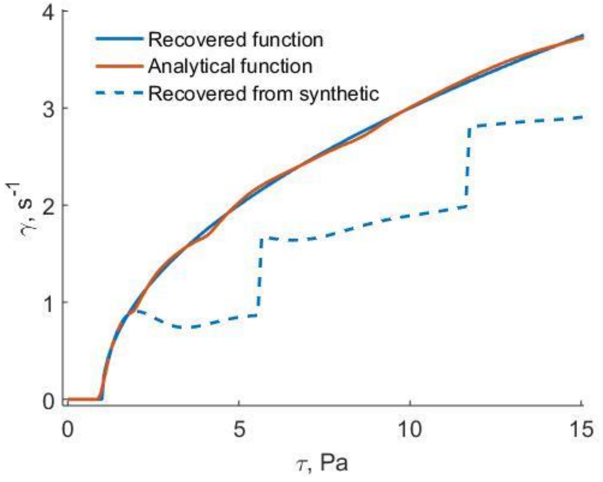
$$\tau = \tau_0 + \mu\dot{\gamma}^2$$

$$\Omega(\xi) = \begin{cases} \dfrac{\sqrt{\xi - \tau_0 R_i^2}}{R_i} - \dfrac{\sqrt{\xi - \tau_0 R_o^2}}{R_o} + \tau_0 \left[\arcsin\left(R_i\sqrt{\dfrac{\tau_0}{\xi}}\right) - \arcsin\left(R_o\sqrt{\dfrac{\tau_0}{\xi}}\right)\right], & \dfrac{\xi}{\tau_0} \geq R_o^2 \\ \dfrac{\sqrt{\xi - \tau_0 R_i^2}}{R_i} + \tau_0 \left[\arcsin\left(R_i\sqrt{\dfrac{\tau_0}{\xi}}\right) - \dfrac{\pi}{2}\right], & R_i^2 \leq \dfrac{\xi}{\tau_0} < R_o^2 \end{cases}$$

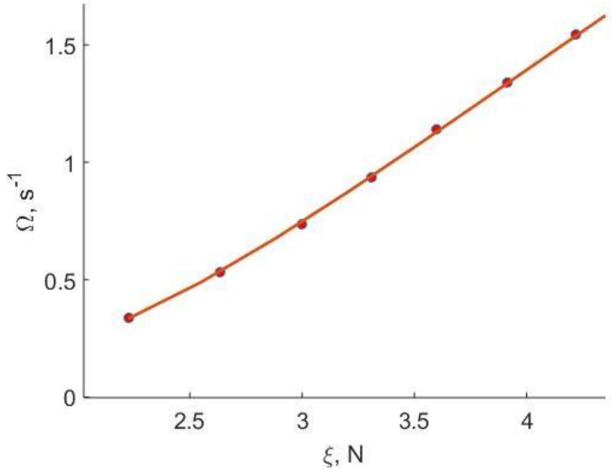
$\tau_0 = 1.0$ Pa, $\mu = 1$ Pa s, $R_i = 0.1$ m, $R_o = 0.145$ m

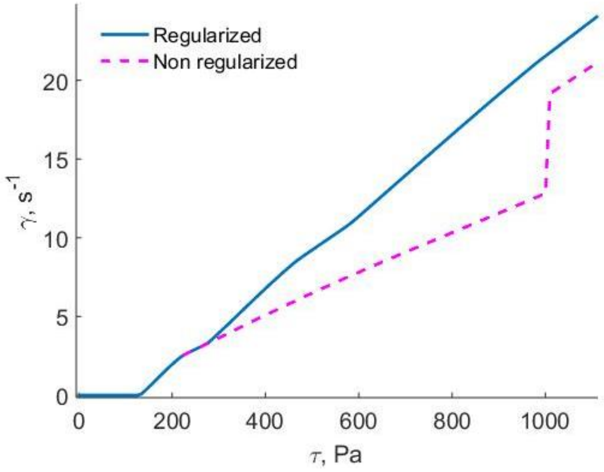




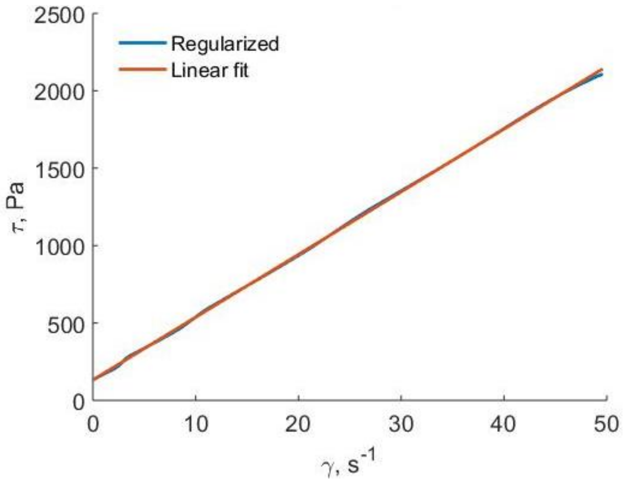


Rheology characterization
Experimental data processing I





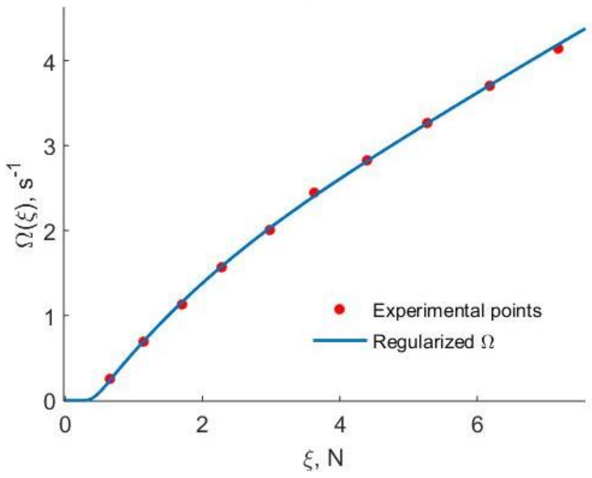
$\tau_0^* = 131$ Pa

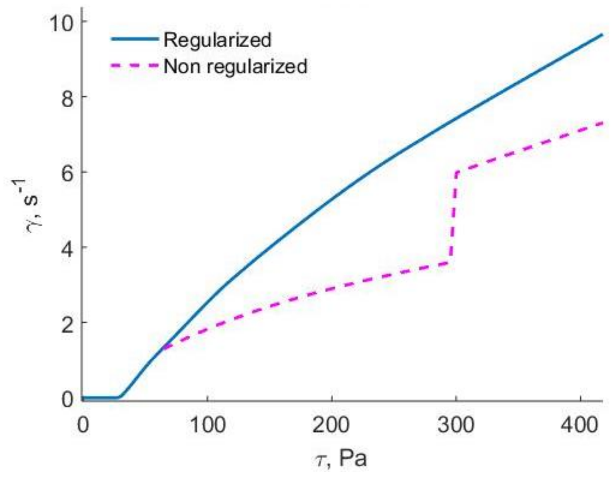


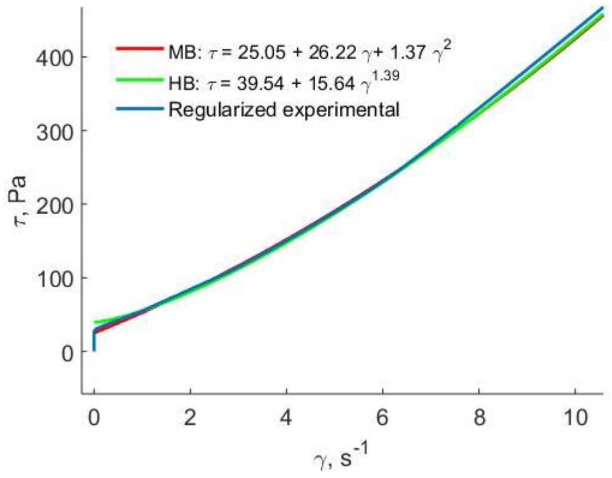
$\tau = \tau_0 + \mu\dot{\gamma}$

$\tau_0 = 125$ Pa

Rheology characterization
Experimental data processing II







τ_0^* = 29 Pa

MB: $\tau = \tau_0 + \mu\dot{\gamma} + c\dot{\gamma}^2$ → 25 Pa

HB: $\tau = \tau_0 + k\dot{\gamma}^n$ → 40 Pa

Concluding remarks

- **Model-independent inversion** framework derived from exact Couette flow mechanics – no constitutive law assumed
- **Unified treatment of yield and non-yield materials,** with direct recovery of shear-rate profiles and yield stress
- **Ill-posed inverse problem stabilized by physically motivated regularization** (data fidelity + smoothness)
- **Validated on synthetic and real concrete data** – accurate yield stress recovery and transparent identifiability limits

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